

A Calculus of Cyber-Physical Systems

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Abstract. We propose a hybrid process calculus for modelling and reasoning on *cyber-physical systems* (CPSs). The dynamics of the calculus is expressed in terms of a *labelled transition system* in the SOS style of Plotkin. This is used to define a *bisimulation-based* behavioural semantics which support compositional reasonings. Finally, we prove run-time properties and system equalities for a non-trivial case study.

Keywords: Process calculus, cyber-physical system, semantics.

1 Introduction

Cyber-Physical Systems (CPSs) are integrations of networking and distributed computing systems with physical processes, where feedback loops allow physical processes to affect computations and vice versa. For example, in real-time control systems, a hierarchy of *sensors*, *actuators* and *control processing components* are connected to control stations. Different kinds of CPSs include supervisory control and data acquisition (SCADA), programmable logic controllers (PLC) and distributed control systems.

The *physical plant* of a CPS is typically represented by means of a *discrete-time state-space model*³ consisting of two equations of the form

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k + e_k\end{aligned}$$

where $x_k \in \mathbb{R}^n$ is the current (*physical*) *state*, $u_k \in \mathbb{R}^m$ is the *input* (i.e., the control actions implemented through actuators) and $y_k \in \mathbb{R}^p$ is the *output* (i.e., the measurements from the sensors). The *uncertainty* $w_k \in \mathbb{R}^n$ and the *measurement error* $e_k \in \mathbb{R}^p$ represent perturbation and sensor noise, respectively, and A , B , and C are matrices modelling the dynamics of the physical system. The *next state* x_{k+1} depends on the current state x_k and the corresponding control actions u_k , at the sampling instant $k \in \mathbb{N}$. Note that, the state x_k cannot be directly observed: only its measurements y_k can be observed.

The physical plant is supported by a communication network through which the sensor measurements and actuator data are exchanged with the *controller(s)*, i.e., the *cyber* component, also called logics, of a CPS (see Figure 1).

³ See [20] for a taxonomy of time-scale models used to represent CPSs.

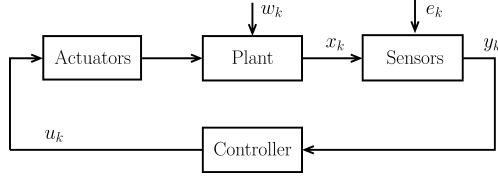


Fig. 1. Structure of a CPS

The range of CPSs applications is rapidly increasing and already covers several domains [10]: advanced automotive systems, energy conservation, environmental monitoring, avionics, critical infrastructure control (electric power, water resources, and communications systems for example), etc.

However, there is still a lack of research on the modelling and validation of CPSs through formal methodologies that might allow to model the interactions among the system components, and to verify the correctness of a CPS, as a whole, before its practical implementation. A straightforward utilisation of these techniques is for *model-checking*, i.e. to statically assess whether the current system deployment can guarantee the expected behaviour. However, they can also be an important aid for system planning, for instance to decide whether different deployments for a given application are behavioural equivalent.

In this paper, we propose a contribution in the area of *formal methods* for CPSs, by defining a *hybrid process calculus*, called CCPS, with a clearly-defined *behavioural semantics* for specifying and reasoning on CPSs. In CCPS, systems are represented as terms of the form $E \bowtie P$, where E denotes the *physical plant* (also called environment) of the system, containing information on state variables, actuators, sensors, evolution law, etc., while P represents the *cyber component of the system*, i.e., the *controller* that governs sensor reading and actuator writing, as well as channel-based communication with other cyber components. Thus, channels are used for logical interactions between cyber components, whereas sensors and actuators make possible the interaction between cyber and physical components. Despite this conceptual similarity, messages transmitted via channels are “consumed” upon reception, whereas actuators’ states (think of a valve) remains unchanged until its controller modifies it.

CCPS is equipped with a *labelled transition semantics* (LTS) in the SOS style of Plotkin [17]. We prove that our labelled transition semantics satisfies some standard time properties such as: *time determinism*, *patience*, *maximal progress*, and *well-timedness*. Based on our LTS, we define a natural notion of *weak bisimilarity*. As a main result, we prove that our bisimilarity is a congruence and it is hence suitable for *compositional reasoning*. We are not aware of similar results in the context of CPSs. Finally, we provide a non-trivial *case study*, taken from an engineering application, and use it to illustrate our definitions and our semantic theory for CPSs. Here, we wish to remark that while we have kept the example simple, it is actually far from trivial and designed to show that various CPSs can be modelled in this style.

Outline In § 2, we give syntax and operational semantics of CCPS. In § 3 we provide a bisimulation-based behavioural semantics for CCPS and prove its compositionality. In § 4 we model in CCPS our case study, and prove for it run-time properties as well as system equalities. In § 5, we discuss related and future work.

2 The Calculus

In this section, we introduce our *Calculus of Cyber-Physical Systems* CCPS. Let us start with some preliminary notations. We use $x, x_k \in \mathcal{X}$ for *state variables*; $c, d \in \mathcal{C}$ for *communication channels*, $a, a_k \in \mathcal{A}$ for *actuator devices*, $s, s_k \in \mathcal{S}$ for *sensors devices*. *Actuator names* are metavariables for actuator devices like *valve*, *light*, etc. Similarly, *sensor names* are metavariables for sensor devices, e.g., a sensor *thermometer* that measures, with a given precision, a state variable called *temperature*. *Values*, ranged over by $v, v' \in \mathcal{V}$, are built from basic values, such as Booleans, integers and real numbers; they also include names.

Given a generic set of names \mathcal{N} , we write $\mathbb{R}^{\mathcal{N}}$ to denote the set of functions assigning a real value to each name in \mathcal{N} . For $\xi \in \mathbb{R}^{\mathcal{N}}$, $n \in \mathcal{N}$ and $v \in \mathbb{R}$, we write $\xi[n \mapsto v]$ to denote the function $\psi \in \mathbb{R}^{\mathcal{N}}$ such that $\psi(m) = \xi(m)$, for any $m \neq n$, and $\psi(n) = v$. For $\xi, \xi' \in \mathbb{R}^{\mathcal{N}}$, we write $\xi \leq \xi'$ if $\xi(x) \leq \xi'(x)$, for any $x \in \mathcal{N}$. Given $\xi_1 \in \mathbb{R}^{\mathcal{N}_1}$ and $\xi_2 \in \mathbb{R}^{\mathcal{N}_2}$ such that $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, we denote with $\xi_1 \uplus \xi_2$ the function in $\mathbb{R}^{\mathcal{N}_1 \cup \mathcal{N}_2}$ such that $(\xi_1 \uplus \xi_2)(x) = \xi_1(x)$, if $x \in \mathcal{N}_1$, and $(\xi_1 \uplus \xi_2)(x) = \xi_2(x)$, if $x \in \mathcal{N}_2$. Finally, given $\xi \in \mathbb{R}^{\mathcal{N}}$ and a set of names $\mathcal{M} \subseteq \mathcal{N}$, we write $\xi|_{\mathcal{M}}$ for the restriction of function ξ to the set \mathcal{M} .

In CCPS, a *cyber-physical system* consists of two components: a *physical environment* E that encloses all physical aspects of a system (state variables, physical devices, evolution law, etc) and a *cyber component*, represented as a concurrent process P that interacts with the physical devices (sensors and actuators) of the system, and can communicate, via channels, with other processes of the same CPS or with processes of other CPSs.

We write $E \bowtie P$ to denote the resulting CPS, and use M and N to range over CPSs. Let us formally define physical environments.

Definition 1 (Physical environment). Let $\hat{\mathcal{X}} \subseteq \mathcal{X}$ be a set of state variables, $\hat{\mathcal{A}} \subseteq \mathcal{A}$ be a set of actuators, and $\hat{\mathcal{S}} \subseteq \mathcal{S}$ be a set of sensors. A physical environment E is 7-tuple $\langle \xi_x, \xi_u, \xi_w, \text{evol}, \xi_e, \text{meas}, \text{inv} \rangle$, where:

- $\xi_x \in \mathbb{R}^{\hat{\mathcal{X}}}$ is the state function,
- $\xi_u \in \mathbb{R}^{\hat{\mathcal{A}}}$ is the actuator function,
- $\xi_w \in \mathbb{R}^{\hat{\mathcal{X}}}$ is the uncertainty function,
- $\text{evol} : \mathbb{R}^{\hat{\mathcal{X}}} \times \mathbb{R}^{\hat{\mathcal{A}}} \times \mathbb{R}^{\hat{\mathcal{X}}} \rightarrow 2^{\mathbb{R}^{\hat{\mathcal{X}}}}$ is the evolution map,
- $\xi_e \in \mathbb{R}^{\hat{\mathcal{S}}}$ is the sensor-error function,
- $\text{meas} : \mathbb{R}^{\hat{\mathcal{X}}} \times \mathbb{R}^{\hat{\mathcal{S}}} \rightarrow 2^{\mathbb{R}^{\hat{\mathcal{S}}}}$ is the measurement map,
- $\text{inv} : \mathbb{R}^{\hat{\mathcal{X}}} \rightarrow \{\text{true}, \text{false}\}$ is the invariant function.

All the functions defining an environment are total functions.

The *state function* ξ_x returns the current value (in \mathbb{R}) associated to each state variable of the system. The *actuator function* ξ_u returns the current value associated to each actuator. The *uncertainty function* ξ_w returns the uncertainty associated to each state variable. Thus, given a state variable $x \in \hat{\mathcal{X}}$, $\xi_w(x)$ returns the maximum distance between the real value of x and its representation in the model. Both the state function and the actuator function are supposed to change during the evolution of the system, whereas the uncertainty function is supposed to be constant.

Given a state function, an actuator function, and an uncertainty function, the *evolution map* $evol$ returns the set of next *admissible state functions*. This function models the *evolution law* of the physical system, where changes made on actuators may reflect on state variables. Since we assume an uncertainty in our models, the evolution map does not return a single state function but a set of possible state functions. The evolution map is obviously monotone with respect to uncertainty: if $\xi_w \leq \xi'_w$ then $evol(\xi_x, \xi_u, \xi_w) \subseteq evol(\xi_x, \xi_u, \xi'_w)$. Note also that, although the uncertainty function is constant, it can be used in the evolution map in an arbitrary way (e.g., it could have a heavier weight when a state variable reaches extreme values).

The *sensor-error function* ξ_e returns the maximum error associated to each sensor in $\hat{\mathcal{S}}$. Again due to the presence of the sensor-error function, the *measurement map* $meas$, given the current state function, returns a set of admissible measurement functions rather than a single one.

Finally, the *invariant function* inv represents the conditions that the state variables must satisfy to allow for the evolution of the system. A CPS whose state variables don't satisfy the invariant is in *deadlock*.

Let us now formalise in **CCPS** the cyber components of CPSs. Our (logical) processes build on the *timed process algebra TPL* [9] (basically CCS enriched with a discrete notion of time). We extend TPL with two constructs: one to read values detected at sensors, and one to write values on actuators. The remaining processes of the calculus are the same as those of TPL.

Definition 2 (Processes). Processes are defined by the grammar:

$$P, Q ::= \text{nil} \mid \text{idle}.P \mid P \parallel Q \mid [\pi.P]Q \mid [b]\{P\}, \{Q\} \mid P \setminus c \mid X \mid \text{rec } X.P.$$

We write nil for the *terminated process*. The process $\text{idle}.P$ sleeps for one time unit and then continues as P . We write $P \parallel Q$ to denote the *parallel composition* of concurrent processes P and Q . The process $[\pi.P]Q$, with $\pi \in \{\text{snd } c\langle v \rangle, \text{rcv } c(x), \text{read } s(x), \text{write } a\langle v \rangle\}$, denotes *prefixing with timeout*. Thus, $[\text{snd } c\langle v \rangle.P]Q$ sends the value v on channel c and, after that, it continues as P ; otherwise, if no communication partner is available within one time unit, it evolves into Q . The process $[\text{rcv } c(x).P]Q$ is the obvious counterpart for receiving. $[\text{read } s(x).P]Q$ reads the value v detected by the sensor s and, after that, it continues as P , where x is replaced by v ; otherwise, after one time unit, it evolves into Q . $[\text{write } a\langle v \rangle.P]Q$ writes the value v on the actuator a and, after that, it continues as P ; otherwise, after one time unit, it evolves into Q . The process $P \setminus c$ is the channel restriction operator of CCS. It is quantified over the

set \mathcal{C} of communication channels but we often use the shorthand $P \setminus C$ to mean $P \setminus c_1 \setminus c_2 \cdots \setminus c_n$, for $C = \{c_1, c_2, \dots, c_n\}$. The process $[b]\{P\}, \{Q\}$ is the standard conditional, where b is a decidable guard. For simplicity, as in CCS, we identify process $[b]\{P\}, \{Q\}$ with P , if b evaluates to true, and $[b]\{P\}, \{Q\}$ with Q , if b evaluates to false. In processes of the form $\text{idle}.Q$ and $[\pi.P]Q$, the occurrence of Q is said to be *time-guarded*. The process $\text{rec } X.P$ denotes *time-guarded recursion* as all occurrences of the process variable X may only occur time-guarded in P .

In the two constructs $[\text{rcv } c(x).P]Q$ and $[\text{read } s(x).P]Q$, the variable x is said to be *bound*. Similarly, the process variable X is bound in $\text{rec } X.P$. This gives rise to the standard notions of *free/bound (process) variables* and α -conversion. We identify processes up to α -conversion (similarly, we identify CPSs up to renaming of state variables, sensor names, and actuator names). A term is *closed* if it does not contain free (process) variables, and we assume to always work with closed processes: the absence of free variables is preserved at run-time. As further notation, we write $T\{v/x\}$ for the substitution of the variable x with the value v in any expression T of our language. Similarly, $T\{P/X\}$ is the substitution of the process variable X with the process P in T .

The syntax of our CPSs is slightly too permissive as a process might use sensors and/or actuators which are not defined in the physical environment.

Definition 3 (Well-formedness). *Given a process P and an environment $E = \langle \xi_x, \xi_u, \xi_w, \text{evol}, \xi_e, \text{meas}, \text{inv} \rangle$, the CPS $E \bowtie P$ is well-formed if: (i) for any sensor s mentioned in P , the function ξ_e is defined in s ; (ii) for any actuator a mentioned in P , the function ξ_u is defined in a .*

Hereafter, we will always work with well-formed networks.

Finally, we assume a number of *notational conventions*. We write $\pi.P$ instead of $\text{rec } X. [\pi.P]X$, when X does not occur in P . We write $\text{snd } c$ (resp. $\text{rcv } c$) when channel c is used for pure synchronisation. For $k \geq 0$, we write $\text{idle}^k.P$ as a shorthand for $\text{idle}.\text{idle} \dots \text{idle}.P$, where the prefix idle appears k consecutive times. Given $M = E \bowtie P$, we write $M \parallel Q$ for $E \bowtie (P \parallel Q)$, and $M \setminus c$ for $E \bowtie P \setminus c$.

2.1 Labelled Transition Semantics

In this section, we provide the dynamics of CCPS in terms of a *labelled transition system (LTS)* in the SOS style of Plotkin. In Definition 4, for convenience, we define some auxiliary operators on environments.

Definition 4. *Let $E = \langle \xi_x, \xi_u, \xi_w, \text{evol}, \xi_e, \text{meas}, \text{inv} \rangle$.*

- $\text{read_sensor}(E, s) = \{\xi(s) : \xi \in \text{meas}(\xi_x, \xi_e)\},$
- $\text{update_act}(E, a, v) = \langle \xi_x, \xi_u[a \mapsto v], \xi_w, \text{evol}, \xi_e, \text{meas}, \text{inv} \rangle,$
- $\text{next}(E) = \bigcup_{\xi \in \text{evol}(\xi_x, \xi_u, \xi_w)} \{\langle \xi, \xi_u, \xi_w, \text{evol}, \xi_e, \text{meas}, \text{inv} \rangle\},$
- $\text{inv}(E) = \text{inv}(\xi_x).$

The operator $\text{read_sensor}(E, s)$ returns the set of possible measurements detected by sensor s in the environment E ; it returns a set of possible values rather than a single value due to the error $\xi_e(s)$ of sensor s . $\text{update_act}(E, a, v)$ returns the

(Outp) $\frac{-}{[\text{snd } c\langle v \rangle.P]Q \xrightarrow{\bar{c}v} P}$	(Inpp) $\frac{-}{[\text{rcv } c(x).P]Q \xrightarrow{cv} P\{v/x\}}$
(Write) $\frac{-}{[\text{write } a\langle v \rangle.P]Q \xrightarrow{a!v} P}$	(Read) $\frac{-}{[\text{read } s(x).P]Q \xrightarrow{s?v} P\{v/x\}}$
(Com) $\frac{P \xrightarrow{\bar{c}v} P' \quad Q \xrightarrow{cv} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'}$	(Par) $\frac{P \xrightarrow{\lambda} P' \quad \lambda \neq \text{idle}}{P \parallel Q \xrightarrow{\lambda} P' \parallel Q}$
(ChnRes) $\frac{P \xrightarrow{\lambda} P' \quad \lambda \notin \{cv, \bar{c}v\}}{P \setminus c \xrightarrow{\lambda} P' \setminus c}$	(Rec) $\frac{P\{\text{rec } X.P/X\} \xrightarrow{\lambda} Q}{\text{rec } X.P \xrightarrow{\lambda} Q}$
(TimeNil) $\frac{-}{\text{nil} \xrightarrow{\text{idle}} \text{nil}}$	(Delay) $\frac{-}{\text{idle}.P \xrightarrow{\text{idle}} P}$
(Timeout) $\frac{-}{[\pi.P]Q \xrightarrow{\text{idle}} Q}$	(TimePar) $\frac{P \xrightarrow{\text{idle}} P' \quad Q \xrightarrow{\text{idle}} Q' \quad P \parallel Q \not\xrightarrow{\tau}}{P \parallel Q \xrightarrow{\text{idle}} P' \parallel Q'}$

Table 1. LTS for processes

new environment in which the actuator function is updated in such a manner to associate the actuator a with the value v . $\text{next}(E)$ returns the set of the next admissible environments reachable from E , by an application of the evolution map. $\text{inv}(E)$ checks whether the state variables satisfy the invariant (here, with an abuse of notation, we overload the meaning of the function inv).

In Table 1, we provide transition rules for processes. Here, the meta-variable λ ranges over labels in the set $\{\text{idle}, \tau, \bar{c}v, cv, a!v, s?v\}$. Rules (Outp), (Inpp) and (Com) serve to model channel communication, on some channel c . Rules (Write) denotes the writing of some data v on an actuator a . Rule (Read) denotes the reading of some data v via a sensor s . Rule (Par) propagates untimed actions over parallel components. Rules (ChnRes) and (Rec) are the standard rules for channel restriction and recursion, respectively. The following four rules are standard, and model the passage of one time unit. The symmetric counterparts of rules (Com) and (Par) are obvious and thus omitted from the table.

In Table 2, we lift the transition rules from processes to systems. All rules have a common premise $\text{inv}(E)$: a CPS can evolve only if the invariant is satisfied, otherwise it is deadlocked. Here, actions, ranged over by α , are in the set $\{\tau, \bar{c}v, cv, \text{idle}\}$. These actions denote: non-observable activities (τ); observable logical activities, *i.e.*, channel transmission ($\bar{c}v$ and cv); the passage of time (idle). Rules (Out) and (Inp) model transmission and reception, with an external system, on a channel c . Rule (SensRead) models the reading of the current data detected at sensor s . Rule (ActWrite) models the writing of a value v on an actuator a . Rule (Tau) lifts non-observable actions from processes to systems. A similar lifting occurs in rule (Time) for timed actions, where $\text{next}(E)$ returns the set of possible

$$\begin{array}{c}
\text{(Out)} \quad \frac{P \xrightarrow{\bar{c}v} P' \quad \text{inv}(E)}{E \bowtie P \xrightarrow{\bar{c}v} E \bowtie P'} \qquad \text{(Inp)} \quad \frac{P \xrightarrow{cv} P' \quad \text{inv}(E)}{E \bowtie P \xrightarrow{cv} E \bowtie P'} \\
\\
\text{(SensRead)} \quad \frac{P \xrightarrow{s?v} P' \quad \text{inv}(E) \quad v \in \text{read_sensor}(E, s)}{E \bowtie P \xrightarrow{\tau} E \bowtie P'} \\
\\
\text{(ActWrite)} \quad \frac{P \xrightarrow{a!v} P' \quad \text{inv}(E) \quad E' = \text{update_act}(E, a, v)}{E \bowtie P \xrightarrow{\tau} E' \bowtie P'} \\
\\
\text{(Tau)} \quad \frac{P \xrightarrow{\tau} P' \quad \text{inv}(E)}{E \bowtie P \xrightarrow{\tau} E \bowtie P'} \quad \text{(Time)} \quad \frac{P \xrightarrow{\text{idle}} P' \quad E \bowtie P \not\xrightarrow{\tau} \text{inv}(E) \quad E' \in \text{next}(E)}{E \bowtie P \xrightarrow{\text{idle}} E' \bowtie P'}
\end{array}$$

Table 2. LTS for CPSs

environments for the next time slot. Thus, by an application of rule (Time) a CPS moves to the next physical state, in the next time slot.

Now, having defined the actions that can be performed by a CPS, we can easily concatenate these actions to define *execution traces*. Formally, given a trace $t = \alpha_1 \dots \alpha_n$, we will write \xrightarrow{t} as an abbreviation for $\xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n}$.

Below, we report a few desirable time properties which hold in our calculus: (a) *time determinism*, (b) *maximal progress*, (c) *patience*, and (d) *well-timedness* (symbol \equiv denotes standard *structural congruence* for timed processes [15,14]).

Theorem 5 (Time properties). *Let $M = E \bowtie P$.*

- (a) *If $M \xrightarrow{\text{idle}} \hat{E} \bowtie Q$ and $M \xrightarrow{\text{idle}} \tilde{E} \bowtie R$, then $\{\hat{E}, \tilde{E}\} \subseteq \text{next}(E)$ and $Q \equiv R$.*
- (b) *If $M \xrightarrow{\tau} M'$ then there is no M'' such that $M \xrightarrow{\text{idle}} M''$.*
- (c) *If $M \xrightarrow{\text{idle}} M'$ for no M' then either $\text{next}(E) = \emptyset$ or $\text{inv}(M) = \text{false}$ or there is N such that $M \xrightarrow{\tau} N$.*
- (d) *For any M there is a $k \in \mathbb{N}$ such that if $M \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} N$, with $\alpha_i \neq \text{idle}$, then $n \leq k$.*

Well-timedness [14,5] ensures the absence of infinite instantaneous traces which would prevent the passage of time, and hence the physical evolution of a CPS.

3 Bisimulation

Once defined the labelled transition semantics, we are ready to define our bisimulation-based behavioural equality for CPSs. We recall that the only *observable activities* in CCPS are: time passing and channel communication. As a consequence, the capability to observe physical events depends on the capability of the cyber components to recognise those events by acting on sensors and actuators, and then signalling them using (unrestricted) channels.

We adopt a standard notation for weak transitions: we write \Rightarrow for the reflexive and transitive closure of τ -actions, namely $(\xrightarrow{\tau})^*$, whereas $\xRightarrow{\alpha}$ means $\Rightarrow \xrightarrow{\alpha} \Rightarrow$, and finally $\xRightarrow{\hat{\alpha}}$ denotes \Rightarrow if $\alpha = \tau$ and $\xRightarrow{\alpha}$ otherwise.

Definition 6 (Bisimulation). *A binary symmetric relation \mathcal{R} over CPSs is a bisimulation if $M \mathcal{R} N$ and $M \xrightarrow{\alpha} M'$ implies that there exists N' such that $N \xRightarrow{\hat{\alpha}} N'$ and $M' \mathcal{R} N'$. We say that M and N are bisimilar, written $M \approx N$, if $M \mathcal{R} N$ for some bisimulation \mathcal{R} .*

A main result of the paper is that our bisimilarity can be used to compare CPSs in a compositional manner. In particular, our bisimilarity is preserved by parallel composition of (non-interfering) CPSs, by parallel composition of (non-interfering) processes, and by channel restriction.

Two CPSs do not interfere with each other if they have a disjoint physical plant. Thus, let $E^i = \langle \xi_x^i, \xi_u^i, \xi_w^i, \text{evol}^i, \xi_e^i, \text{meas}^i, \text{inv}^i \rangle$ with sensors in $\hat{\mathcal{S}}_i$, actuators in $\hat{\mathcal{A}}_i$, and state variables in $\hat{\mathcal{X}}_i$, for $i \in \{1, 2\}$. If $\hat{\mathcal{S}}_1 \cap \hat{\mathcal{S}}_2 = \emptyset$ and $\hat{\mathcal{A}}_1 \cap \hat{\mathcal{A}}_2 = \emptyset$ and $\hat{\mathcal{X}}_1 \cap \hat{\mathcal{X}}_2 = \emptyset$, then we define the *disjoint union* of the environments E_1 and E_2 , written $E_1 \uplus E_2$, to be the environment $\langle \xi_x, \xi_u, \xi_w, \text{evol}, \xi_e, \text{meas}, \text{inv} \rangle$ such that: $\xi_x = \xi_x^1 \uplus \xi_x^2$, $\xi_u = \xi_u^1 \uplus \xi_u^2$, $\xi_w = \xi_w^1 \uplus \xi_w^2$, $\xi_e = \xi_e^1 \uplus \xi_e^2$, and

$$\begin{aligned} \text{evol}(\xi, \psi, \phi) &= \{\xi' = \xi_1 \uplus \xi_2 : \xi_i \in \text{evol}^i(\xi|_{\hat{\mathcal{X}}_i}, \psi|_{\hat{\mathcal{A}}_i}, \phi|_{\hat{\mathcal{X}}_i}), \text{ for } i \in \{1, 2\}\} \\ \text{meas}(\xi, \psi) &= \{\xi' = \xi_1 \uplus \xi_2 : \xi_i \in \text{meas}^i(\xi|_{\hat{\mathcal{X}}_i}, \psi|_{\hat{\mathcal{S}}_i}), \text{ for } i \in \{1, 2\}\} \\ \text{inv}(\xi) &= \text{inv}^1(\xi|_{\hat{\mathcal{X}}_1}) \wedge \text{inv}^2(\xi|_{\hat{\mathcal{X}}_2}) . \end{aligned}$$

Definition 7 (Non-interfering CPSs). *Let $M_i = E_i \bowtie P_i$, for $i \in \{1, 2\}$. We say that M_1 and M_2 do not interfere with each other if E_1 and E_2 have disjoint sets of state variables, sensors and actuators. In this case, we write $M_1 \uplus M_2$ to denote the CPS defined as $(E_1 \uplus E_2) \bowtie (P_1 \parallel P_2)$.*

A similar but simpler definition can be given for processes. Let $M = E \bowtie P$, a non-interfering process Q is a process which does not interfere with the plant E as it never accesses its sensors and/or actuators. Thus, in the system $M \parallel Q$ the process Q cannot interfere with the physical evolution of M . However, process Q can definitely affect the observable behaviour of the whole system by communicating on channels. Notice that, as we only consider well-formed CPSs (Definition 3), a non-interfering processes is basically a (pure) TPL process [9].

Definition 8 (Non-interfering processes). *A process P is called non-interfering if it never acts on sensors and/or actuators.*

Now, everything is in place to prove the compositionality of our bisimilarity \approx .

Theorem 9 (Congruence results). *Let M and N be two CPSs.*

1. $M \approx N$ implies $M \uplus O \approx N \uplus O$, for any non-interfering CPS O
2. $M \approx N$ implies $M \parallel P \approx N \parallel P$, for any non-interfering process P
3. $M \approx N$ implies $M \setminus c \approx N \setminus c$, for any channel c .

The presence of invariants in the definition of physical environment makes the proof of the second item of the theorem above non standard.

As we will see in the next section, these compositional properties will be very useful when reasoning about complex systems.

4 Case study

In this section, we model in CCPS an engine, called *Eng*, whose temperature is maintained within a specific range by means of a cooling system. The physical environment *Env* of the engine is constituted by: (i) a state variable *temp* containing the current temperature of the engine; (ii) an actuator *cool* to turn on/off the cooling system; (iii) a sensor s_t (such as a thermometer or a thermocouple) measuring the temperature of the engine; (iv) an uncertainty $\delta = 0.4$ associated to the only variable *temp*; (v) a simple evolution law that increases (resp., decreases) the value of *temp* of one degree per time unit if the cooling system is inactive (resp., active) — the evolution law is obviously affected by the uncertainty δ ; (vi) an error $\epsilon = 0.1$ associated to the only sensor s_t ; (vii) a measurement map to get the values detected by sensor s_t , up to its error ϵ ; (viii) an invariant function saying that the system gets faulty when the temperature of the engine gets out of the range $[0, 30]$.

Formally, $Env = \langle \xi_x, \xi_u, \xi_w, evol, \xi_e, meas, inv \rangle$ with:

- $\xi_x \in \mathbb{R}^{\{temp\}}$ and $\xi_x(temp) = 0$;
- $\xi_u \in \mathbb{R}^{\{cool\}}$ and $\xi_u(cool) = \text{off}$; for the sake of simplicity, we can assume ξ_u to be a mapping $\{cool\} \rightarrow \{\text{on}, \text{off}\}$ such that $\xi_u(cool) = \text{off}$ if $\xi_u(cool) \geq 0$, and $\xi_u(cool) = \text{on}$ if $\xi_u(cool) < 0$;
- $\xi_w \in \mathbb{R}^{\{temp\}}$ and $\xi_w(temp) = 0.4 = \delta$;
- $evol(\xi_x^i, \xi_u^i, \xi_w) = \{ \xi : \xi(temp) = \xi_x^i(temp) + heat(\xi_u^i, cool) + \gamma \wedge \gamma \in [-\delta, +\delta] \}$, where $heat(\xi_u^i, cool) = -1$ if $\xi_u^i(cool) = \text{on}$ (active cooling), and $heat(\xi_u^i, cool) = +1$ if $\xi_u^i(cool) = \text{off}$ (inactive cooling);
- $\xi_e \in \mathbb{R}^{\{s_t\}}$ and $\xi_e(s_t) = 0.1 = \epsilon$;
- $meas(\xi_x, \xi_e) = \{ \xi : \xi(s_t) \in [\xi_x^i(temp) - \epsilon, \xi_x^i(temp) + \epsilon] \}$;
- $inv(\xi_x) = \text{true}$ if $0 \leq \xi_x(temp) \leq 30$; $inv(\xi_x) = \text{false}$, otherwise.

The cyber component of *Eng* consists of a process *Ctrl* which models the controller activity. Intuitively, process *Ctrl* senses the temperature of the engine at each time interval. When the sensed temperature is above 10, the controller activates the coolant. The cooling activity is maintained for 5 consecutive time units. After that time, if the temperature does not drop below 10 then the controller transmits its *ID* on a specific channel for signalling a *warning*, it keeps cooling for another 5 time units, and then checks again the sensed temperature; otherwise, if the sensed temperature is not above the threshold 10, the controller turns off the cooling and moves to the next time interval. Formally,

$$\begin{aligned}
 Ctrl &= \text{rec } X. \text{read } s_t(x). [x > 10] \{ Cooling \}, \{ \text{idle}.X \} \\
 Cooling &= \text{write } cool\langle \text{on} \rangle. \text{rec } Y. \text{idle}^5. \text{read } s_t(x). \\
 &\quad [x > 10] \{ \text{snd } warning\langle ID \rangle.Y \}, \{ \text{write } cool\langle \text{off} \rangle. \text{idle}.X \} .
 \end{aligned}$$

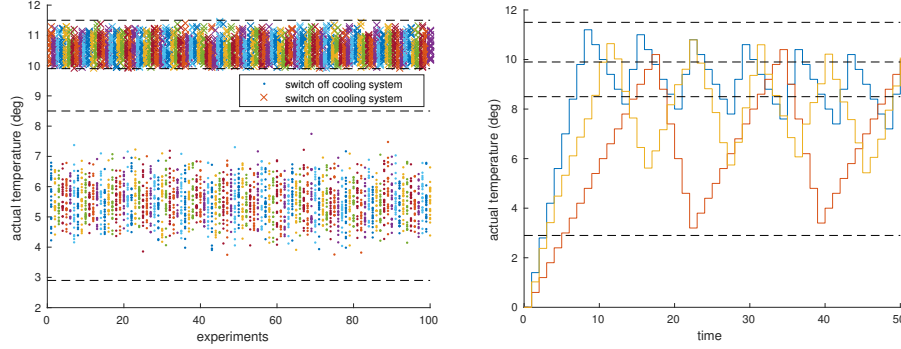


Fig. 2. Simulations in MATLAB of the engine *Eng*

The whole engine is defined as: $Eng = Env \bowtie Ctrl$, where *Env* is the physical environment defined before.

Our operational semantics allows us to formally prove a number of *run-time properties* of our engine. For instance, the following proposition says that our engine never reaches a warning state and never deadlocks. never reaches a warning state.

Proposition 10. *Let Eng be the CPS defined before. If $Eng \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} Eng'$, for some Eng' , then $\alpha_i \in \{\tau, idle\}$, for $1 \leq i \leq n$, and there is Eng'' such that $Eng' \xrightarrow{\alpha} Eng''$, for some $\alpha_i \in \{\tau, idle\}$.*

Actually, we can be quite precise on the temperature reached by the engine before and after the cooling activity: in each of the 5 time slots of cooling, the temperature will drop of a value laying in the interval $[1-\delta, 1+\delta]$, where δ is the uncertainty of the model. Formally,

Proposition 11. *For any execution trace of Eng , we have:*

- when *Eng* turns on the cooling, the value of the state variable *temp* ranges over $(10 - \epsilon, 11 + \epsilon + \delta]$;
- when *Eng* turns off the cooling, the value of the variable *temp* ranges over $(10 - \epsilon - 5*(1+\delta), 11 + \epsilon + \delta - 5*(1-\delta)]$.

In Figure 2, the left graphic collects a campaign of 100 simulations, lasting 250 time units each, showing that the value of the state variable *temp* when the cooling system is turned on (resp., off) lays in the interval $(9.9, 11.5]$ (resp., $(2.9, 8.5]$); these bounds are represented by the dashed horizontal lines. Since $\delta = 0.4$, these results are in line with those of Proposition 11. The right graphic shows three examples of possible evolutions in time of the state variable *temp*.

Now, the reader may wonder whether it is possible to design a variant of our engine which meets the same specifications with better performances. For instance, an engine consuming less coolant. Let us consider a variant of the engine described before:

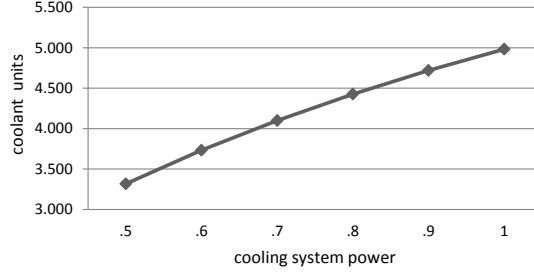


Fig. 3. Simulations in MATLAB of coolant consumption

$$\overline{Eng} = \overline{Env} \bowtie Ctrl .$$

Here, \overline{Env} is the same as Env except for the evolution map, as we set $heat(\xi_u^i, cool) = -0.8$ if $\xi_u^i(cool) = \text{on}$. This means that in \overline{Eng} we reduce the power of the cooling system by 20%. In Figure 3, we report the results of our simulations over 10000 runs lasting 10000 time units each. From this graph, \overline{Eng} saves in average more than 10% of coolant with respect to Eng . So, the new question is: are these two engines behavioural equivalent? Do they meet the same specifications?

Our bisimilarity provides us with a precise answer to these questions.

Proposition 12. *The two variants of the engine are bisimilar: $Eng \approx \overline{Eng}$.*

At this point, one may wonder whether it is possible to improve the performances of our engine even more. For instance, by reducing the power of the cooling system by a further 10%, by setting $heat(\xi_u^i, cool) = -0.7$ if $\xi_u^i(cool) = \text{on}$. We can formally prove that this is not the case.

Proposition 13. *Let \widehat{Eng} be the same as Eng , except for the evolution map, in which $heat(\xi_u^i, cool) = -0.7$ if $\xi_u^i(cool) = \text{on}$. Then, $Eng \not\approx \widehat{Eng}$.*

Finally, we show how we can use the compositionality of our behavioural semantics (Theorem 9) to deal with bigger CPSs. Suppose that Eng denotes the modelisation of an airplane engine. In this case, we could define in CCPS a very simple *airplane control system* that checks whether the left engine (Eng_L) and the right engine (Eng_R) are signalling warnings. The whole CPS is defined as follows:

$$Airplane = ((Eng_L \uplus Eng_R) \parallel Check) \setminus \{warning\}$$

where $Eng_L = Eng\{^L/ID\}\{^{temp-l}/temp\}\{^{cool-l}/cool\}\{^{st-l}/st\}$, and, similarly, $Eng_R = Eng\{^R/ID\}\{^{temp-r}/temp\}\{^{cool-r}/cool\}\{^{st-r}/st\}$, and process $Check$ is defined as:

$$\begin{aligned} Check &= \text{rec } X. [\text{rcv } warning(x). [x = L] \{ Check_1^L, \{ Check_1^R \} \}] X \\ Check_i^{id} &= [\text{rcv } warning(y). [y \neq id] \{ \text{snd } alarm.idle.X \}, \{ idle.Check_{i+1}^{id} \}] Check_{i+1}^{id} \\ Check_5^{id} &= [\text{rcv } warning(z). [z \neq id] \{ \text{snd } alarm.idle.X \}, \{ \text{snd } failure\langle id \rangle.idle.X \}] \\ &\quad \text{snd } failure\langle id \rangle.X \end{aligned}$$

for $1 \leq i \leq 5$. Intuitively, if one of the two engines is in a warning state then the process $Check_i^{id}$, for $id \in \{L, R\}$, checks whether also the second engine moves into a warning state, in the following 5 time intervals (i.e. during the cooling cycle). If both engines gets in a warning state then an *alarm* is sent, otherwise, if only one engine is facing a warning then the airplane control system yields a *failure* signalling which engine is not working properly.

So, since we know that $Eng \approx \overline{Eng}$, the final question becomes the following: can we safely equip our airplane with the more performant engines, $\overline{Eng_L}$ and $\overline{Eng_R}$, in which $heat(\xi_u^i, cool) = -0.8$ if $\xi_u^i(cool) = \text{on}$, without affecting the whole observable behaviour of the airplane? The answer is “yes”, and this result can be formally proved by applying Proposition 12 and Theorem 9.

Proposition 14. *Let $\overline{Airplane} = ((\overline{Eng_L} \uplus \overline{Eng_R}) \parallel Check) \setminus \{warning\}$. Then, $Airplane \approx \overline{Airplane}$.*

5 Related and Future Work

A number of approaches have been proposed for modelling CPSs using formal methods. For instance, *hybrid automata* [1] combine finite state transition systems with discrete variables (whose values capture the state of the modelled discrete or cyber components) and continuous variables (whose values capture the state of the modelled continuous or physical components).

Hybrid process algebras [6] are a powerful tool for reasoning about physical systems and provide techniques for analysing and verifying protocols for hybrid automata. CCPS shares some similarities with the ϕ -calculus [18], a hybrid extension of the π -calculus [15]. In the ϕ -calculus, a hybrid system is represented as a pair (E, P) , where E is the environment and P is the process interacting with the environment. Unlike CCPS, in ϕ -calculus, given a system (E, P) the process P can dynamically change both the evolution law and the invariant of the system. However, the ϕ -calculus does not have a representation of physical devices and measurement law. Concerning behavioural semantics, the ϕ -calculus is equipped with a weak bisimilarity between systems that is not compositional.

In the HYPE process algebra [8], the continuous part of the system is represented by appropriate variables whose changes are determined by active influences (i.e., commands on actuators). The authors defines a strong bisimulation that extends the *ic-bisimulation* of [3]. Unlike ic-bisimulation, the bisimulation in HYPE is preserved by a notion of parallel composition that is slightly more permissive than ours. However, bisimilar systems in HYPE must always have the same influence. Thus, in HYPE we cannot compare CPSs sending different commands on actuators at the same time, as we do in Proposition 12.

Vigo et al. [19] proposed a calculus for wireless-based cyber-physical systems endowed with a theory to study cryptographic primitives, together with explicit notions of communication failure and unwanted communication. The calculus does not provide any notion of behavioural equivalence. It also lacks a clear distinction between physical and logical components.

Lanese et al. [11] proposed an untimed calculus of mobile IoT devices interacting with the physical environment by means of sensors and actuators. The calculus does not allow any representation of the physical environment, and the bisimilarity is not preserved by parallel composition (compositionality is recovered by significantly strengthening the discriminating power).

Lanotte and Merro [12] extended and generalised the work of [11] in a timed setting by providing a bisimulation-based semantic theory that is suitable for compositional reasoning. As in [11], the physical environment is not represented.

Bodei et al. [4] proposed an untimed process calculus supporting a control flow analysis to track how data spread from sensors to the logics of the network, and how physical data are manipulated. Sensors and actuators are modelled as value-passing CCS channels. The dynamics of the calculus is given in terms of a reduction relation and no behavioural equivalence is defined.

As regards future works, we believe that our paper can lay and streamline *theoretical foundations* for the development of formal and automated tools to verify CPSs before their practical implementation. To that end, we will consider applying, possibly after proper enhancements, existing tools and frameworks for automated verification, such as Maude [16], Ariadne [2], and SMC UPPAAL [7], resorting to the development of an dedicated tool if existing ones prove not up to the task. Finally, in [13], we developed an extended version of **CCPS** to provide a formal study of a variety of *cyber-physical attacks* targeting physical devices. Also in this case, the final goal is to develop formal and automated tools to analyse security properties of CPSs.

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A Proofs

We recall that the cyber-components our CPSs are basically TPL-processes [9] extended with constructs to read sensors and write actuators. TPL already enjoys time determinism, patience and maximal progress. The well-timedness property is present in many process calculi with a discrete notion of time (e.g. [14]) similar to ours. Thus, it is straightforward to rewrite the proofs of those results for our slight variant of TPL.

Proposition 15 (Processes time properties [9,14]).

- If $P \xrightarrow{\text{idle}} Q$ and $P \xrightarrow{\text{idle}} R$, then $Q \equiv R$.
- If $P \xrightarrow{\tau} P'$ then there is no P'' such that $P \xrightarrow{\text{idle}} P''$.
- If $P \xrightarrow{\text{idle}} P'$ for no P' then there is Q such that $P \xrightarrow{\tau} Q$.
- For any P there is a $k \in \mathbb{N}$ such that if $P \xrightarrow{\lambda_1} \dots \xrightarrow{\lambda_n} P$, with $\lambda_i \neq \text{idle}$, then $n \leq k$.

The challenge in the proof of Theorem 5 is to lift the results of Proposition 15 to the CPSs of CCPS.

In its standard formulation, *time determinism* says that a system reaches at most one new state by executing a *idle*-action. However, by an application

of Rule (Time), our CPSs may nondeterministically move into a new physical environment, according to the evolution law.

Proposition 16 (Time determinism for CPSs). *If $M \xrightarrow{\text{idle}} \hat{E} \bowtie Q$ and $M \xrightarrow{\text{idle}} \tilde{E} \bowtie R$, then $\{\hat{E}, \tilde{E}\} \subseteq \text{next}(E)$ and $Q \equiv R$.*

Proof. Let $M = E \bowtie P$. Since the only possible rule to derive $\xrightarrow{\text{idle}}$ is rule (Time), then we have that there is Q, R, \hat{E}, \tilde{E} such that

$$\frac{P \xrightarrow{\text{idle}} Q \quad M \not\xrightarrow{\tau} \text{inv}(E) \quad \hat{E} \in \text{next}(E)}{M \xrightarrow{\text{idle}} \hat{E} \bowtie Q}$$

and

$$\frac{P \xrightarrow{\text{idle}} R \quad M \not\xrightarrow{\tau} \text{inv}(E) \quad \tilde{E} \in \text{next}(E)}{M \xrightarrow{\text{idle}} \tilde{E} \bowtie R}.$$

The result follows by Proposition 15.

According to [9], the *maximal progress* property says that processes communicate as soon as a possibility of communication arises. In our calculus, we generalise this property saying that τ -actions cannot be delayed, independently on how they are generated.

Proposition 17 (Maximal progress for CPSs). *If $M \xrightarrow{\tau} M'$ then there is no M'' such that $M \xrightarrow{\text{idle}} M''$.*

Proof. The proof is by contradiction. Let us suppose $M \xrightarrow{\text{idle}} M''$, for some M'' . This is only possible by an application of rule (Time):

$$\frac{P \xrightarrow{\text{idle}} P' \quad M \not\xrightarrow{\tau} \text{inv}(E) \quad E' \in \text{next}(E)}{M \xrightarrow{\text{idle}} M''}$$

with $M'' = E' \bowtie P'$. However, the premises requires $M \not\xrightarrow{\tau}$ which contradicts the fact that $M \xrightarrow{\tau} M'$.

Patience in CCPS is more involved with respect to the same property in TPL. It basically says that if a CPS cannot evolve in time, then either (i) the physical plant does not contemplate an evolution, or (ii) the invariant is violated, or (iii) the CPS can perform an internal action.

Proposition 18 (Patience for CPS). *If $M \xrightarrow{\text{idle}} M'$ for no M' then either $\text{next}(E) = \emptyset$ or $\text{inv}(M) = \text{false}$ or there is N such that $M \xrightarrow{\tau} N$.*

Proof. The proof is by contradiction. Let us suppose that $M \xrightarrow{\text{idle}} M'$ for no M' , and $M \not\xrightarrow{\tau}$ and $\text{inv}(E) = \text{true}$ and $E' \in \text{next}(E)$, for some E' . Since the only possible rule to derive $\xrightarrow{\text{idle}}$ is rule (Time), then $M \xrightarrow{\text{idle}} M'$ for no M' , implies that the following derivation is not admissible for any P' and E' :

$$\frac{P \xrightarrow{\text{idle}} P' \quad M \not\xrightarrow{\tau} \quad \text{inv}(E) \quad E' \in \text{next}(E)}{M \xrightarrow{\text{idle}} E' \bowtie P'}. \quad .$$

Since $M \not\xrightarrow{\tau}$ and $\text{inv}(E) = \text{true}$ and $E' \in \text{next}(E)$, for some E' , the only possibility is $P \xrightarrow{\text{idle}} P'$ for no P' . Since $P \xrightarrow{\text{idle}} P'$ for no P' , by Proposition 15 we have that $P \xrightarrow{\tau} P'$. Since $\text{inv}(E) = \text{true}$, by an application of rule (Tau) there is N such that $M \xrightarrow{\tau} N$. This contradicts the initially hypothesis that $M \not\xrightarrow{\tau}$.

The following property is well-timedness. It basically says that time passing cannot be prevented by infinite sequences of internal actions.

Proposition 19 (Well-timedness for CPSs). *For any M there is a $k \in \mathbb{N}$ such that if $M \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} N$, with $\alpha_i \neq \text{idle}$, then $n \leq k$.*

Proof. The proof is by contradiction. Suppose there is no k satisfying the statement above. Hence there exists an unbounded derivation

$$M = M_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} M_{n+1} \xrightarrow{\alpha_{n+1}} \dots$$

with $M_i = E_i \bowtie P_i$ and $\alpha_i \neq \text{idle}$.

By inspection of rules of Table 2 we have that, for any i , $M_i \xrightarrow{\alpha_i} M_{i+1}$ and $\alpha_i \neq \text{idle}$ implies that $P_i \xrightarrow{\lambda_i} P_{i+1}$, for some $\lambda_i \neq \text{idle}$. Hence we have the following unbounded derivation

$$P_1 \xrightarrow{\lambda_1} \dots \xrightarrow{\lambda_n} P_{n+1} \xrightarrow{\lambda_{n+1}} \dots$$

with $\lambda_i \neq \text{idle}$. In contradiction with Proposition 15.

Proof of Theorem 5

Proof. The result follows by an application of Proposition 16, Proposition 17, Proposition 18 and Proposition 19.

In order to prove the compositionality of our bisimilarity, i.e. Theorem 9, we divide its statement in three different propositions.

In order to prove that \approx preserves contextuality, we need a number of technical lemmas. Lemma 20 formalises a number of properties of the compound environment $E_1 \uplus E_2$.

Lemma 20. *Let E_1 and E_2 be two physical environments. If defined, the environment $E_1 \uplus E_2$ has the following properties:*

1. $read_sensor(E_1 \uplus E_2, s)$ is equal to $read_sensor(E_1, s)$, if s is a sensor of E_1 , and it is equal to $read_sensor(E_2, s)$, if s is a sensor of E_2 ;
2. $v \in read_sensor(E_1, s)$ implies that $v \in read_sensor(E_1 \uplus E_2, s)$ for any sensor s in E_1 and for any environment E_2 ;
3. $update_act(E_1 \uplus E_2, a, v)$ is equal to $update_act(E_1, a, v) \uplus E_2$, if a is an actuator of E_1 , and it is equal to $E_1 \uplus update_act(E_2, a, v)$, if a is an actuator of E_2 ;
4. $update_act(E_1, a, v) \uplus E_2$ is equal to $update_act(E_1 \uplus E_2, a, v)$ for any actuator a in E_1 and for any environment E_2 ;
5. $next(E_1 \uplus E_2) = \{ E'_1 \uplus E'_2 : E'_1 \in next(E_1) \text{ and } E'_2 \in next(E_2) \}$;
6. $inv(E_1 \uplus E_2) = inv(E_1) \wedge inv(E_2)$.

Proof. By definition of the operator \uplus on physical environments.

Lemma 21 serves to propagate untimed actions on parallel CPSs.

Lemma 21. *If $M \xrightarrow{\alpha} M'$, with $\alpha \neq \text{idle}$, then $M \uplus O \xrightarrow{\alpha} M' \uplus O$, for any non-interfering CPS O , with $inv(O) = \text{true}$.*

Proof. The proof is by rule induction on why $M \xrightarrow{\alpha} M'$. Let us suppose that $M = E_1 \bowtie P_1$ and $O = E_2 \bowtie P_2$, for some E_1, E_2, P_1 and P_2 . We can distinguish several cases on why $M \xrightarrow{\alpha} M'$. We prove the case in which $M \xrightarrow{\alpha} M'$ is derived by an application of rule (SensRead). The other cases can be proved in a similar manner. In this case, we have $\alpha = \tau$ and there are s, v , and P'_1 such that

$$\frac{P_1 \xrightarrow{s?v} P'_1 \quad inv(E_1) \quad v \in read_sensor(E_1, s)}{M \xrightarrow{\tau} M'}$$

with $M' = E_1 \bowtie P'_1$.

Since $P_1 \xrightarrow{s?v} P'_1$, by an applicaiton of rule (Par) we can derive $P_1 \parallel P_2 \xrightarrow{s?v} P'_1 \parallel P_2$. Since $inv(E_1) = \text{true}$ and, by hypothesis, $inv(E_2) = \text{true}$, by an application of Lemma 20(6) we derive that $inv(E_1 \uplus E_2) = \text{true}$. Since $v \in read_sensor(E_1, s)$, by an application of Lemma 20(2) we derive that $v \in read_sensor(E_1 \uplus E_2, s)$. This is enough to derive that:

$$\frac{P_1 \parallel P_2 \xrightarrow{s?v} P'_1 \parallel P_2 \quad inv(E_1 \uplus E_2) \quad v \in read_sensor(E_1 \uplus E_2, s)}{M \uplus O \xrightarrow{\tau} E_1 \uplus E_2 \bowtie P'_1 \parallel P_2}.$$

Hence the result follows by assuming $M' = E_1 \bowtie P'$ and $M' \uplus O = E_1 \uplus E_2 \bowtie P'_1 \parallel P_2$.

Next lemma says the invariants of bisimilar CPSs must agree.

Lemma 22. *$M \approx N$ implies $inv(M) = inv(N)$.*

Proof. The proof is by contradiction. Suppose that $M \approx N$, $inv(M) = \text{true}$ and $inv(N) = \text{false}$ (the other case is similar). By Proposition 19, there exists a finite derivation $M \xrightarrow{\tau} M_1 \xrightarrow{\tau} \dots \xrightarrow{\tau} M_n$, with $M_n \not\xrightarrow{\tau}$. Since $inv(N) = \text{false}$, the CPSs N cannot perform any action, and in particular, $N \not\xrightarrow{\tau}$. From $M \approx N$ we derive that $M_i \approx N$, for $1 \leq i \leq n$. Since $M_n \not\xrightarrow{\tau}$, by Proposition 17 it follows that $M_n \xrightarrow{\text{idle}} M'$, for some M' . Since $inv(N) = \text{false}$, we have $N \not\xrightarrow{\text{idle}}$ and hence also $N \not\Rightarrow$.

Summarising: $M_n \approx N$, $M_n \xrightarrow{\text{idle}} M'$ and $N \not\Rightarrow$ which contradict the definition of bisimilarity.

Here comes one of the main technical result: the bisimilarity is preserved by the parallel composition of non-interfering CPSs.

Proposition 23. *$M \approx N$ implies $M \uplus O \approx N \uplus O$, for any non-interfering CPS O .*

Proof. We show that the relation $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is a bisimulation where:

$$\begin{aligned} \mathcal{R}_1 &= \{(M \uplus O, N \uplus O) : M \approx N\} \\ \mathcal{R}_2 &= \{(M, N) : inv(M) = inv(N) = \text{false}\} . \end{aligned}$$

The relation \mathcal{R}_2 is trivially a bisimulation because it contains pairs of deadlocked CPSs. Thus, we focus on when $(M \uplus O, N \uplus O) \in \mathcal{R}_1$.

We proceed by case analysis on why $M \uplus O \xrightarrow{\alpha} \hat{M}$ (the case when $N \uplus O \xrightarrow{\alpha} \hat{N}$ is symmetric).

- Let $M \uplus O \xrightarrow{\tau} \hat{M}$, with $M = E_1 \bowtie P_1$ and $O = E_2 \bowtie P_2$, for some E_1, E_2, P_1 and P_2 , by an application of rule (SensRead). This implies that

$$\frac{P_1 \parallel P_2 \xrightarrow{s?v} P' \quad inv(E_1 \uplus E_2) \quad v \in read_sensor(E_1 \uplus E_2, s)}{M \uplus O \xrightarrow{\tau} \hat{M}}$$

with $\hat{M} = E_1 \uplus E_2 \bowtie P'$. We recall that by definition of \uplus the environments E_1 and E_2 have different physical devices. Thus, there are two cases:

- s is a sensor of E_1 .

In this case, $P_1 \xrightarrow{s?v} P'_1$, for some P'_1 , and hence $P' = P'_1 \parallel P_2$. Since $inv(E_1 \uplus E_2) = \text{true}$ and $v \in read_sensor(E_1 \uplus E_2, s)$, by an application of Lemma 20(1) and Lemma 20(6), we derive $inv(E_1) = inv(E_2) = \text{true}$ and $v \in read_sensor(E_1, s)$. Now, let $M' = E_1 \bowtie P'_1$; it follows that $\hat{M} = M' \uplus O$. Since $P_1 \xrightarrow{s?v} P'_1$, $inv(E_1) = \text{true}$, and $v \in read_sensor(E_1, s)$, by an application of rule (SensRead) we have $M \xrightarrow{\tau} M'$. As $M \approx N$, there is N' such that $N \Rightarrow N'$ with $M' \approx N'$. Since $inv(E_2) = \text{true}$, by several applications of Lemma 21 it follows that $N \uplus O \Rightarrow N' \uplus O = \hat{N}$, with $(\hat{M}, \hat{N}) \in \mathcal{R}_1 \subseteq \mathcal{R}$.

- s is a sensor of E_2 .

In this case, $P_2 \xrightarrow{s?v} P'_2$, for some P'_2 , and hence $P' = P_1 \parallel P'_2$. Let $O' = E_2 \bowtie P'_2$; it follows that $\hat{M} = M \uplus O' = E_1 \uplus E_2 \bowtie (P_1 \parallel P'_2)$. Let $N = E_3 \bowtie P_3$, for some E_3 and P_3 . By an application of rule (Par) we have that $P_3 \parallel P_2 \xrightarrow{s?v} P_3 \parallel P'_2$. Since $\text{inv}(E_1 \uplus E_2)$ and $v \in \text{read_sensor}(E_1 \uplus E_2, s)$, by an application of Lemma 20(1) and Lemma 20(6), we derive $\text{inv}(E_1) = \text{inv}(E_2) = \text{true}$ and $v \in \text{read_sensor}(E_2, s)$. As $M \approx N$, by Lemma 22 it follows that $\text{inv}(E_3) = \text{true}$, and hence $\text{inv}(E_3 \uplus E_2) = \text{true}$. Since $v \in \text{read_sensor}(E_2, s)$, by Lemma 20(2), it follows that $v \in \text{read_sensor}(E_3 \uplus E_2, s)$.

Summarising $P_3 \parallel P_2 \xrightarrow{s?v} P_3 \parallel P'_2$, $v \in \text{read_sensor}(E_3 \uplus E_2, s)$, and $\text{inv}(E_3 \uplus E_2) = \text{true}$. Thus, by an application of rule (SensRead) we have $N \uplus O \xrightarrow{\tau} N \uplus O'$, with $(M \uplus O', N \uplus O') \in \mathcal{R}_1 \subseteq \mathcal{R}$.

- Let $M \uplus O \xrightarrow{\tau} \hat{M}$, with $M = E_1 \bowtie P_1$ and $O = E_2 \bowtie P_2$, for some E_1 , E_2 , P_1 and P_2 , by an application of rule (ActWrite). This case is similar to the previous ones. Basically we apply Lemma 20(3) instead of Lemma 20(1), and Lemma 20(4) instead of Lemma 20(2).
- Let $M \uplus O \xrightarrow{\tau} \hat{M}$, with $M = E_1 \bowtie P_1$ and $O = E_2 \bowtie P_2$, for some E_1 , E_2 , P_1 and P_2 , by an application of rule (Tau):

$$\frac{P_1 \parallel P_2 \xrightarrow{\tau} P' \quad \text{inv}(E_1 \uplus E_2)}{M \uplus O \xrightarrow{\tau} \hat{M}}$$

with $\hat{M} = E_1 \uplus E_2 \bowtie P'$. We can distinguish four cases.

- Let $P_1 \parallel P_2 \xrightarrow{\tau} P'$ by an application of rule (Par), because $P_1 \xrightarrow{\tau} P'_1$ and $P' = P'_1 \parallel P_2$, for some P'_1 . Since $\text{inv}(E_1 \uplus E_2)$, by Lemma 20(6), $\text{inv}(E_1) = \text{inv}(E_2) = \text{true}$. Let $M' = E_1 \bowtie P'_1$; we have that $\hat{M} = M' \uplus O$. Since $P_1 \xrightarrow{\tau} P'_1$ and $\text{inv}(E_1) = \text{true}$, by an application of rule (Tau) we derive $M \xrightarrow{\tau} M'$. As $M \approx N$, there is N' such that $N \Rightarrow N'$ with $M' \approx N'$. Since $\text{inv}(E_2) = \text{true}$, by several applications of Lemma 21 we have that $N \uplus O \Rightarrow N' \uplus O = \hat{N}$, with $(\hat{M}, \hat{N}) \in \mathcal{R}_1 \subseteq \mathcal{R}$.
- Let $P_1 \parallel P_2 \xrightarrow{\tau} P'$ by an application of rule (Par), because $P_2 \xrightarrow{\tau} P'_2$ and $P' = P_1 \parallel P'_2$, for some P'_2 . Let $O' = E_2 \bowtie P'_2$; it follows that $\hat{M} = M \uplus O' = E_1 \uplus E_2 \bowtie (P_1 \parallel P'_2)$. Let $N = E_3 \bowtie P_3$, for some E_3 and P_3 . By an application of rule (Par) we have that $P_3 \parallel P_2 \xrightarrow{\tau} P_3 \parallel P'_2$. Since $\text{inv}(E_1 \uplus E_2)$, by an application of Lemma 20(6), we derive $\text{inv}(E_1) = \text{inv}(E_2) = \text{true}$. As $M \approx N$, by Lemma 22 it follows that $\text{inv}(E_3) = \text{true}$, and hence $\text{inv}(E_3 \uplus E_2) = \text{true}$. Summarising $P_3 \parallel P_2 \xrightarrow{\tau} P_3 \parallel P'_2$ and $\text{inv}(E_3 \uplus E_2) = \text{true}$. Thus, by an application of rule (Tau) we have $N \uplus O \xrightarrow{\tau} N \uplus O' = \hat{N}$, with $(\hat{M}, \hat{N}) \in \mathcal{R}_1 \subseteq \mathcal{R}$.
- Let $P_1 \parallel P_2 \xrightarrow{\tau} P'$ by an application of rule (Com) because $P_1 \xrightarrow{\bar{c}v} P'_1$ and $P_2 \xrightarrow{cv} P'_2$ and $P' = P'_1 \parallel P'_2$, for some P'_1 and P'_2 . Since $\text{inv}(E_1 \uplus$

$E_2) = \text{true}$, by Lemma 20(6) follows that $\text{inv}(E_1) = \text{inv}(E_2) = \text{true}$. Let $M' = E_1 \bowtie P'_1$ and $O' = E_2 \bowtie P'_2$; we have that $\hat{M} = M' \uplus O'$. Since $P_1 \xrightarrow{\bar{c}v} P'_1$ and $\text{inv}(E_1) = \text{true}$, by an application of rule (Out) we have $M \xrightarrow{\bar{c}v} M'$. As $M \approx N$, there are N_1, N_2 and N' such that $N \Rightarrow N_1 \xrightarrow{\bar{c}v} N_2 \Rightarrow N'$, with $M' \approx N'$. Since $\text{inv}(E_2) = \text{true}$, by an appropriate number of applications of Lemma 21 we have that $N \uplus O \Rightarrow N_1 \uplus O$. Moreover, $N_1 \xrightarrow{\bar{c}v} N_2$ implies that $P_3 \xrightarrow{\bar{c}v} P'_3$ for some P_3 and P'_3 and E_3 such that $N_1 = E_3 \bowtie P_3$ and $N_2 = E_3 \bowtie P'_3$ and $\text{inv}(E_3) = \text{true}$. Since $P_2 \xrightarrow{cv} P'_2$ we can use rules (Com) to derive $P_2 \parallel P_3 \xrightarrow{\tau} P'_2 \parallel P'_3$. Moreover from the fact that both $\text{inv}(E_2) = \text{true}$ and $\text{inv}(E_3) = \text{true}$ we can derive, by Lemma 20(6), that $\text{inv}(E_3 \uplus E_2) = \text{true}$.

Summarising $P_2 \parallel P_3 \xrightarrow{\tau} P'_2 \parallel P'_3$ and $\text{inv}(E_3 \uplus E_2) = \text{true}$, and, for $O = E_2 \bowtie P_2$ and $O' = E_2 \bowtie P'_2$, we can use rule (Tau) to derive $N_1 \uplus O \xrightarrow{\tau} N_2 \uplus O'$. Since $\text{inv}(E_2) = \text{true}$, by an appropriate number of applications of Lemma 21, we get $N_2 \uplus O' \Rightarrow N' \uplus O'$. As $M' \approx N'$, it follows that $(M' \uplus O', N' \uplus O') \in \mathcal{R}_1 \subseteq \mathcal{R}$.

- Let $P_1 \parallel P_2 \xrightarrow{\tau} P'$ by an application of rule (Com) because $P_1 \xrightarrow{cv} P'_1$ and $P_2 \xrightarrow{\bar{c}v} P'_2$, for some P'_1 and P'_2 . This case is similar to the previous one.
- Let $M \uplus O \xrightarrow{\text{idle}} \hat{M}$, with $M = E_1 \bowtie P_1$ and $O = E_2 \bowtie P_2$, for some E_1, E_2, P_1 and P_2 . This action can be derived only by an application of rule (Time):

$$\frac{P_1 \parallel P_2 \xrightarrow{\text{idle}} P' \quad M \uplus O \not\xrightarrow{\tau} \quad \text{inv}(E_1 \uplus E_2) \quad E' \in \text{next}(E_1 \uplus E_2)}{M \uplus O \xrightarrow{\text{idle}} \hat{M}}$$

with $\hat{M} = E' \bowtie P'$.

The derivation $P_1 \parallel P_2 \xrightarrow{\text{idle}} P'$ follows by an application of rule (TimePar) because $P_1 \xrightarrow{\text{idle}} P'_1$ and $P_2 \xrightarrow{\text{idle}} P'_2$, for some P'_1 and P'_2 , such that $P' = P'_1 \parallel P'_2$. Since $\text{inv}(E_1 \uplus E_2)$, by Lemma 20(6) follows that $\text{inv}(E_1) = \text{inv}(E_2) = \text{true}$. Since $E' \in \text{next}(E_1 \uplus E_2)$, by Lemma 20(5) follows that $E' = E'_1 \uplus E'_2$, for some $E'_1 \in \text{next}(E_1)$ and $E'_2 \in \text{next}(E_2)$. Furthermore, since $M \uplus O \not\xrightarrow{\tau}$, by Lemma 21 follows that $M \not\xrightarrow{\tau}$ and $O \not\xrightarrow{\tau}$.

Thus, since $P_1 \xrightarrow{\text{idle}} P'_1$, $M \not\xrightarrow{\tau}$, $\text{inv}(E_1) = \text{true}$, and $E'_1 \in \text{next}(E_1)$, by an application of rule (Time) it follows that $M \xrightarrow{\text{idle}} M'$, with $M' = E'_1 \bowtie P'_1$. Similarly, from $P_2 \xrightarrow{\text{idle}} P'_2$, $O \not\xrightarrow{\tau}$, $\text{inv}(E_2) = \text{true}$, and $E'_2 \in \text{next}(E_2)$, we can derive that $O \xrightarrow{\text{idle}} O'$, with $O' = E'_2 \bowtie P'_2$. As a consequence, $\hat{M} = M' \uplus O' = E'_1 \uplus E'_2 \bowtie P'_1 \parallel P'_2$.

Now, from $M \approx N$ and $M \xrightarrow{\text{idle}} M'$, there are N_1, N_2 and N' such that $N \Rightarrow N_1 \xrightarrow{\text{idle}} N_2 \Rightarrow N'$, with $M' \approx N'$. Since $\text{inv}(E_2) = \text{true}$, by an appropriate number of applications of Lemma 21 we have that $N \uplus O \Rightarrow N_1 \uplus O$. Next, we

show that we can apply rule (Time) to derive $N_1 \uplus O \xrightarrow{\text{idle}} N_2 \uplus O'$. For that we only need to prove that $N_1 \uplus O \not\xrightarrow{\tau}$. We reason by contradiction. Since $M \approx N$ and $N \Rightarrow N_1$, there is M_1 such that $M \Rightarrow M_1$, with $M_1 \approx N_1$. Since $M \not\xrightarrow{\tau}$, it follows that $M = M_1 \approx N_1$. Since $N_1 \xrightarrow{\text{idle}} N_2$ and $O \xrightarrow{\text{idle}} O'$, by an application of Theorem 5(b) we can derive $N_1 \not\xrightarrow{\tau}$ and $O \not\xrightarrow{\tau}$. Thus, $N_1 \uplus O \xrightarrow{\tau}$ could be derived only by an application of rule (Com) where N_1 interact with O , via some channel c . However, as $N_1 \approx M$ the network M could mimic the same interaction (via the same channel c) with O , giving rise to a reduction of the form $M \uplus O \Rightarrow \xrightarrow{\tau}$. This is in contradiction with the initial premises that $M \uplus O \not\xrightarrow{\tau}$. Thus, $N_1 \uplus O \not\xrightarrow{\tau}$ and by an application of rule (Time) we can derive $N_1 \uplus O \xrightarrow{\text{idle}} N_2 \uplus O'$. It remains to determine the possible evolutions of $N_2 \uplus O'$.

There are two cases:

- The invariant of O' is true.

In this case, by an appropriate number of applications of Lemma 21 we get $N_2 \uplus O' \Rightarrow N' \uplus O'$. Thus, $N \uplus O \xrightarrow{\text{idle}} N' \uplus O'$, with $(M' \uplus O', N' \uplus O') \in \mathcal{R}_1 \subseteq \mathcal{R}$, because $M' \approx N'$.

- The invariant of O' is false.

In this case, we have that $N \uplus O \xrightarrow{\text{idle}} N_2 \uplus O'$, with $(M' \uplus O', N' \uplus O') \in \mathcal{R}_2 \subseteq \mathcal{R}$, because $\text{inv}(O') = \text{false}$.

- Let $M \uplus O \xrightarrow{cv} \hat{M}$, with $M = E_1 \bowtie P_1$ and $O = E_2 \bowtie P_2$, for some E_1, E_2, E', P_1, P_2 and P' . This derivation can be only due to an application of rule (Inp):

$$\frac{P_1 \parallel P_2 \xrightarrow{cv} P' \quad \text{inv}(E_1 \uplus E_2)}{M \uplus O \xrightarrow{cv} \hat{M}}$$

with $\hat{M} = E_1 \uplus E_2 \bowtie P'$. We distinguish two cases.

- $P_1 \xrightarrow{cv} P'_1$, for some P'_1 such that $P = P'_1 \parallel P_2$.

Then, let $M' = E_1 \bowtie P'_1$; we have that $\hat{M} = M' \uplus O$. Since $\text{inv}(E_1 \uplus E_2)$, by Lemma 20(6), it follows that $\text{inv}(E_1) = \text{inv}(E_2) = \text{true}$. Since $P_1 \xrightarrow{cv} P'_1$ and $\text{inv}(E_1) = \text{true}$, by an application of (Inp) on M we can derive $M \xrightarrow{cv} M'$. As $M \approx N$, there is N' such that $N \xrightarrow{cv} N'$ with $M' \approx N'$. Since $\text{inv}(E_2) = \text{true}$, by several applications of Lemma 21 we have that $N \uplus O \xrightarrow{cv} N' \uplus O = \hat{N}$, with $(\hat{M}, \hat{N}) \in \mathcal{R}_1 \subseteq \mathcal{R}$.

- $P_2 \xrightarrow{cv} P'_2$, for some P'_2 such that $P = P_1 \parallel P'_2$.

Let $O' = E_2 \bowtie P'_2$; we have that $\hat{M} = M \uplus O'$. Since $\text{inv}(E_1 \uplus E_2) = \text{true}$, by Lemma 20(6) it follows that $\text{inv}(E_2) = \text{true}$. Let $N = E_3 \bowtie P_3$, for some E_3 and P_3 . Since $M \approx N$ and $\text{inv}(E_2) = \text{true}$, by Lemma 22 we derive $\text{inv}(E_3) = \text{true}$. By Lemma 20(6) it follows that $\text{inv}(E_3 \uplus E_2) = \text{true}$. Furthermore, by an application of rule (Par) we have $P_3 \parallel P_2 \xrightarrow{cv} P_3 \parallel P'_2$.

Summarising: $P_3 \parallel P_2 \xrightarrow{cv} P_3 \parallel P'_2$ and $\text{inv}(E_3 \uplus E_2) = \text{true}$. Thus, by an application of rule (Inp) we derive $N \uplus O \xrightarrow{cv} N \uplus O' = \hat{N}$, with $(\hat{M}, \hat{N}) \in \mathcal{R}_1 \subseteq \mathcal{R}$.

- Let $M \uplus O \xrightarrow{\bar{c}v} \hat{M}$. This case is similar to the previous one.

Now, let us prove that our bisimilarity is preserved by parallel composition of non-interfering processes. This is a special case of the previous result.

Proposition 24. *$M \approx N$ implies $M \parallel P \approx N \parallel P$, for any non-interfering process P .*

Proof. We have to prove that $M \approx N$ implies $M \parallel P \approx N \parallel P$, for any process P which does not access any physical device.

Let E_\emptyset be the environment with an empty set of state variables, sensors and actuators. It is straightforward to prove that $M \parallel P \approx M \uplus (E_\emptyset \bowtie P)$ and $N \parallel P \approx N \uplus (E_\emptyset \bowtie P)$. Since \approx is preserved by the operator \uplus , the result follows by transitivity of \approx .

Finally, we prove that bisimilarity is preserved by channel restriction.

Proposition 25. *$M \approx N$ implies $M \setminus c \approx N \setminus c$, for any channel c .*

Proof. It is enough to show that the relation $\{(M \setminus c, N \setminus c) : M \approx N\}$ is a bisimulation. The proof proceeds by case analysis on why $M \setminus c \xrightarrow{\alpha} \hat{M}$.

Proof of Theorem 9

Proof. By Proposition 23, Proposition 24 and Proposition 25.

In order to prove Proposition 10 and Proposition 11 we use the following lemma that formalises the invariant properties binding the state variable *temp* with the activity of the cooling system.

Intuitively, when the cooling system is inactive the value of the state variable *temp* lays in the interval $[0, 11 + \epsilon + \delta]$. Furthermore, if the coolant is not active and the variable *temp* lays in the interval $(10 + \epsilon, 11 + \epsilon + \delta]$ then the cooling will be turned on in the next time slot. Finally, when active then cooling system will remain so for $k \in 1..5$ time slots (counting also the current time slot) being the variable *temp* in the real interval $(10 - \epsilon - k*(1+\delta), 11 + \epsilon + \delta - k*(1-\delta)]$.

Lemma 26. *Let Eng be the system defined in § 4. Let*

$$Eng = Eng_1 \xrightarrow{t_1} \xrightarrow{\text{idle}} Eng_2 \xrightarrow{t_2} \xrightarrow{\text{idle}} \dots \xrightarrow{t_{n-1}} \xrightarrow{\text{idle}} Eng_n$$

such that the traces t_j contain no idle-actions, for any $j \in 1..n-1$, and for any $i \in 1..n$ $Eng_i = E_i \bowtie P_i$ with $E_i = \langle \xi_x^i, \xi_u^i, \delta, \text{evol}, \epsilon, \text{meas}, \text{inv} \rangle$. Then, for any $i \in 1..n-1$ we have the following:

1. *if $\xi_u^i(\text{cool}) = \text{off}$ then $\xi_x^i(\text{temp}) \in [0, 11 + \epsilon + \delta]$;*
2. *if $\xi_u^i(\text{cool}) = \text{off}$ and $\xi_x^i(\text{temp}) \in (10 + \epsilon, 11 + \epsilon + \delta]$ then, in the next time slot, $\xi_u^{i+1}(\text{cool}) = \text{on}$;*
3. *if $\xi_u^i(\text{cool}) = \text{on}$ then $\xi_x^i(\text{temp}) \in (10 - \epsilon - k*(1+\delta), 11 + \epsilon + \delta - k*(1-\delta)]$, for some $k \in 1..5$ such that $\xi_u^{i-k}(\text{cool}) = \text{off}$ and $\xi_u^{i-j}(\text{cool}) = \text{on}$, for $j \in 0..k-1$.*

Proof. Let us denote with v_i the values of the state variable *temp* in the systems Eng_i , i.e., $\xi_x^i(temp) = v_i$. Moreover we will say that the coolant is active (resp., is not active) in Eng_i if $\xi_u^i(cool) = \text{on}$ (resp., $\xi_u^i(cool) = \text{off}$).

The proof is by mathematical induction on n , i.e., the number idle-actions of our traces.

The case base $n = 1$ follows directly from the definition of Eng .

Let prove the inductive case. We assume that the three statements holds for $n - 1$ and we prove that they also hold for n .

1. Let us assume that the cooling is not active in Eng_n , then we prove that $v_n \in [0, 11 + \epsilon + \delta]$.

We consider separately the cases in which the coolant is active or not in Eng_{n-1}

- Suppose the coolant is not active in Eng_{n-1} (and inactive in Eng_n).

By inductive hypothesis we have $v_{n-1} \in [0, 11 + \epsilon + \delta]$. Furthermore, if $v_{n-1} \in (10 + \epsilon, 11 + \epsilon + \delta]$ then, by inductive hypotheses, the coolant must be active in Eng_n . Since we know in Eng_n the cooling is not active it follows that $v_{n-1} \in [0, 10 + \epsilon]$. Furthermore, Eng_n the temperature will increase of a value laying in the interval $[1 - \delta, 1 + \delta] = [0.6, 1.4]$. Thus v_n will be in $[0.6, 11 + \epsilon + \delta] \subseteq [0, 11 + \epsilon + \delta]$.

- Suppose the coolant is active in Eng_{n-1} (and inactive in Eng_n).

By inductive hypothesis $v_{n-1} \in (10 - \epsilon - k * (1 + \delta), 11 + \epsilon + \delta - k * (1 - \delta)]$ for some $k \in 1..5$ such that the coolant is not active in Eng_{n-1-k} and is active in $Eng_{n-k}, \dots, Eng_{n-1}$.

The case $k \in \{1, \dots, 4\}$ is not admissible. In fact if $k \in \{1, \dots, 4\}$ then the coolant would be active for less than 5 idle-actions as we know that Eng_n is inactive.

Hence it must be $k = 5$. Since $\delta = 0.4$, $\epsilon = 0.1$ and $k = 5$, it holds that $v_{n-1} \in (10 - 0.1 - 5 * 1.4, 11 + 0.1 + 0.4 - 5 * 0.6] = (2.8, 8.6]$. Moreover, since the coolant is active for 5 idle actions, the controller of Eng_{n-1} checks the temperature. However, since $v_{n-1} \in (2.8, 8.6]$ then the coolant is turned off. Thus, in the next time slot, the temperature will increase of a value in $[1 - \delta, 1 + \delta] = [0.6, 1.4]$. As a consequence in Eng_n we will have $v_n \in [2.8 + 0.6, 8.6 + 1.4] = [3.4, 10] \subseteq [0, 11 + \epsilon + \delta]$.

2. Let us assume that the coolant is not active in Eng_n and $v_n \in (10 + \epsilon, 11 + \epsilon + \delta]$, then we prove that the coolant is active in Eng_{n+1} . Since the coolant is not active in Eng_n then it will check the temperature before the next time slot. Since $v_n \in (10 + \epsilon, 11 + \epsilon + \delta]$ and $\epsilon = 0.1$, then the process *Ctrl* will sense a temperature greater than 10 and the coolant will be turned on. Thus the coolant will be active in Eng_{n+1} .

3. Let us assume that the coolant is active in Eng_n , then we prove that $v_n \in (10 - \epsilon - k * (1 + \delta), 11 + \epsilon + \delta - k * (1 - \delta)]$ for some $k \in 1..5$ and the coolant is not active in Eng_{n-k} and active in $Eng_{n-k+1}, \dots, Eng_n$.

We separate the case in which the coolant is active in Eng_{n-1} from that in which is not active.

- Suppose the coolant is not active in Eng_{n-1} (and active in Eng_n).

In this case $k = 1$ as the coolant is not active in Eng_{n-1} and it is active in Eng_n . Since $k = 1$, we have to prove $v_n \in (10 - \epsilon - (1 + \delta), 11 + \epsilon + \delta - (1 - \delta)]$. However, since the coolant is not active in Eng_{n-1} and is active in Eng_n it means that the coolant has been switched on in Eng_{n-1} because the sensed temperature was above 10 (this may happen only if $v_{n-1} > 10 - \epsilon$). By inductive hypothesis, since the coolant is not active in Eng_{n-1} , we have that $v_{n-1} \in [0, 11 + \epsilon + \delta]$. Therefore, from $v_{n-1} > 10 - \epsilon$ and $v_{n-1} \in [0, 11 + \epsilon + \delta]$ it follows that $v_{n-1} \in (10 - \epsilon, 11 + \epsilon + \delta]$. Furthermore, since the coolant is active in Eng_n , the temperature will decrease of a value in $[1 - \delta, 1 + \delta]$ and therefore $v_n \in (10 - \epsilon - (1 + \delta), 11 + \epsilon + \delta - (1 - \delta)]$ which concludes this case of the proof.

- Suppose the coolant is active in Eng_{n-1} (and active in Eng_n as well). By inductive hypothesis there is $h \in 1..5$ such that $v_{n-1} \in (10 - \epsilon - h * (1 + \delta), 11 + \epsilon + \delta - h * (1 - \delta)]$ and the coolant is not active in Eng_{n-1-h} and is active in $Eng_{n-h}, \dots, Eng_{n-1}$.

The case $h = 5$ is not admissible. In fact, since $\delta = 0.4$ and $\epsilon = 0.1$, if $h = 5$ then $v_{n-1} \in (10 - 0.1 - 5 * 1.4, 11 + 0.1 + \delta - 5 * 0.6] = (2.8, 8.6]$. Furthermore, since the coolant is already active since 5 idle actions, the controller of Eng_{n-1} is supposed to check the temperature. As $v_{n-1} \in (2.8, 8.6]$ the coolant should be turned off. In contradiction with the fact that the coolant is active in Eng_n .

Hence it must be $h \in 1..4$. Let us prove that for $k = h + 1$ we obtain our result. Namely we have to prove that, for $k = h + 1$, (i) $v_n \in (10 - \epsilon - k * (1 + \delta), 11 + \epsilon + \delta - k * (1 - \delta)]$, and (ii) the coolant is not active in Eng_{n-k} and active in $Eng_{n-k+1}, \dots, Eng_n$.

Let us prove the statement (i). By inductive hypotheses, it holds that $v_{n-1} \in (10 - \epsilon - h * (1 + \delta), 11 + \epsilon + \delta - h * (1 - \delta)]$. Since the coolant is active in Eng_n then the temperature will decrease. Hence, $v_n \in (10 - \epsilon - (h + 1) * (1 + \delta), 11 + \epsilon + \delta - (h + 1) * (1 - \delta)]$. Therefore, since $k = h + 1$, we have that $v_n \in (10 - \epsilon - k * (1 + \delta), 11 + \epsilon + \delta - k * (1 - \delta)]$.

Let us prove the statement (ii). By inductive hypothesis the coolant is inactive in Eng_{n-1-h} and it is active in $Eng_{n-h}, \dots, Eng_{n-1}$. Now, since the coolant is active in Eng_n , for $k = h + 1$, we have that the coolant is not active in Eng_{n-k} and is active in $Eng_{n-k+1}, \dots, Eng_n$ which concludes this case of the proof.

Proof of Proposition 10

Proof. By Lemma 26 and since $\delta = 0.4$ and $\epsilon = 0.1$, the value of the state variable *temp* is always in the real interval $[0, 11.5]$. As a consequence, the invariant of the system is never violated and the system never deadlocks. Moreover, after 5 idle -actions of cooling the state variable *temp* is always in the real interval $(10 - 0.1 - 5 * 1.4, 11 + 0.1 + 0.4 - 5 * 0.6] = (2.9, 8.5]$. Hence the process *Ctrl* will never transmit on the channel *warning*.

Proof of Proposition 11

Proof. Let us prove the two statements separately.

- If process *Ctrl* senses a temperature above 10 (and hence *Eng* turns on the cooling) then the value of the state variable *temp* is greater than $10 - \epsilon$. By Lemma 26 the value of the state variable *temp* is always less or equal than $11 + \epsilon + \delta$. Therefore, if *Ctrl* senses a temperature above 10, then the value of the state variable *temp* is in $(10 - \epsilon, 11 + \epsilon + \delta]$.
- By Lemma 26 (third item) the coolant can be active for no more than 5 time slots. Hence, by Lemma 26, when *Eng* turns off the cooling system the state variable *temp* ranges over $(10 - \epsilon - 5 * (1 + \delta), 11 + \epsilon + \delta - 5 * (1 - \delta))$.

In order to prove Proposition 12 we use the following lemma that is a variant of Lemma 26. Differently from Lemma 26, when active then cooling system will remain so for $k \in 1..5$ time slots (counting also the current time slot) being the variable *temp* in the real interval $(10 - \epsilon - k*(0.8 + \delta), 11.5 - k*(0.8 - \delta))$.

Lemma 27. *Let \overline{Eng} be the system defined in § 4. Let*

$$\overline{Eng} = \overline{Eng_1} \xrightarrow{t_1} \xrightarrow{\text{idle}} \overline{Eng_2} \xrightarrow{t_2} \xrightarrow{\text{idle}} \dots \xrightarrow{t_{n-1}} \xrightarrow{\text{idle}} \overline{Eng_n}$$

such that the traces t_j contain no idle-actions, for any $j \in 1..n-1$, and for any $i \in 1..n$ $\overline{Eng_i} = E_i \bowtie P_i$ with $E_i = \langle \xi_x^i, \xi_u^i, \delta, \text{evol}, \epsilon, \text{meas}, \text{inv} \rangle$. Then, for any $i \in 1..n-1$ we have the following:

1. *if $\xi_u^i(\text{cool}) = \text{off}$ then $\xi_x^i(\text{temp}) \in [0, 11 + \epsilon + \delta]$;*
2. *if $\xi_u^i(\text{cool}) = \text{off}$ and $\xi_x^i(\text{temp}) \in (10 + \epsilon, 11 + \epsilon + \delta]$ then, in the next time slot, $\xi_u^{i+1}(\text{cool}) = \text{on}$;*
3. *if $\xi_u^i(\text{cool}) = \text{on}$ then $\xi_x^i(\text{temp}) \in (10 - \epsilon - k*(0.8 + \delta), 11 + \epsilon + \delta - k*(0.8 - \delta)]$, for some $k \in 1..5$ such that $\xi_u^{i-k}(\text{cool}) = \text{off}$ and $\xi_u^{i-j}(\text{cool}) = \text{on}$, for $j \in 0..k-1$.*

Proof. Similar to the proof of Lemma 26.

Proof of Proposition 12

Proof. By Proposition 10 is sufficient to prove that \overline{Eng} has no trace which deadlocks or emits an alarm.

By Lemma 27 and since $\delta = 0.4$ and $\epsilon = 0.1$, the value of the state variable *temp* is always in the real interval $[0, 11.5]$. As a consequence, the invariant of the system is never violated and the system never deadlocks. Moreover, after 5 idle -actions of cooling the state variable *temp* is always in the real interval $(10 - 0.1 - 5 * 1.2, 11 + 0.1 + 0.4 - 5 * 0.4] = (3.9, 9.5]$. Hence the process *Ctrl* will never transmit on the channel *warning*.

Proof of Proposition 13

Proof. It is enough to prove that there exists an execution trace of the engine \widehat{Eng} containing an output along channel *warning*. Then the result follows by an application of Proposition 10.

We can easily build up a trace for \widehat{Eng} in which, after 10 idle-actions, in the 11-th time slot, the value of the state variable *temp* is 10.1. In fact, it is enough to increase the temperature of 1.01 degrees for the first 10 rounds. Notice that

this is an admissible value since, $1.01 \in [1 - \delta, 1 + \delta] = [0.6, 1.4]$. Being 10.1 the value of the state variable *temp*, there is an execution trace in which the sensed temperature is 10 (recall that $\epsilon = 0.1$) and hence the cooling system is not activated. However, in the following time slot, i.e. the 12-th time slot, the temperature may reach at most the value $10.1 + 1 + \delta = 11.5$, imposing the activation of the cooling system. After 5 time units of cooling, in the 17-th time slot, the variable *temp* will be at most $11.5 - 5 * (0.7 - \delta) = 11.5 - 1.5 = 10$. Since $\epsilon = 0.1$, the sensed temperature would be in the real interval $[9.9, 10.1]$. Thus, there is an execution trace in which the sensed temperature is 10.1, which will be greater than 10. As a consequence, the warning will be emitted, in the 17-th time slot.

Proof of Proposition 14

Proof. By Proposition 12 we derive $Eng \approx \overline{Eng}$. By simple α -conversion it follows that $Eng_L \approx \overline{Eng_L}$ and $Eng_R \approx \overline{Eng_R}$, respectively. By Theorem 9(1) (and transitivity of \approx) it follows that $Eng_L \uplus Eng_R \approx \overline{Eng_L} \uplus \overline{Eng_R}$. By Theorem 9(2) it follows that $(Eng_L \uplus Eng_R) \parallel Check \approx (\overline{Eng_L} \uplus \overline{Eng_R}) \parallel Check$. By Theorem 9(3) we obtain $Airplane \approx Airplane$.